

Asymptotic behavior of the loss probability for an M/G/1/N queue with vacations

Yuanyuan Liu*, Central South University
Yiqiang Q. Zhao**, Carleton University

Accepted by *Applied Mathematical Modelling* April-1, 2012

Abstract

In this paper, asymptotic properties of the loss probability are considered for an M/G/1/N queue with server vacations and exhaustive service discipline, denoted by an M/G/1/N-(V, E)-queue. Exact asymptotic rates of the loss probability are obtained for the cases in which the traffic intensity is smaller than, equal to and greater than one, respectively. When the vacation time is zero, the model considered degenerates to the standard M/G/1/N queue. For this standard queueing model, our analysis provides new or extended asymptotic results for the loss probability. In terms of the duality relationship between the M/G/1/N and GI/M/1/N queues, we also provide asymptotic properties for the standard GI/M/1/N model.

Keywords: M/G/1/N queue, GI/M/1/N queue, serve vacations, invariant measure, loss probability, asymptotic behavior

Mathematics Subject Classification (2000): 60K25, 68M20

1 Introduction

For a queueing system with a finite system capacity N , the loss probability $P_{\text{loss}}(N)$, defined as the steady-state probability with which an arriving customer finds no free room in the queue (or buffer) and is forced to leave immediately, is an important performance measure in real applications, see e.g. [1, 4, 10, 16, 21]. Simple closed-form solutions for the loss probability are available only for a very limited number of special cases such as the M/M/c/K+c queue. For others, numerical computations and simulation methods are common tools for the analysis of the loss probability, but they cannot provide closed-form properties. Another important aspect is the asymptotic of the loss probability, which can lead to approximations, performance bounds among others. Also, approximations can be often obtained through asymptotic properties in the invariant measure for the corresponding infinite-buffer system, for which the analysis is usually less challenging than that for the finite-buffer one, see, e.g. [3, 4, 12, 17]. This paper,

*Corresponding author. Postal address: School of Mathematics and Statistics, Railway Campus, Central South University, Changsha, Hunan, 410075, China; Email address: liuyy@csu.edu.cn; Tel.: +86 87312655267; Fax: +86 87315586475.

** Postal address: School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, ON Canada K1S 5B6; E-mail address: zhao@math.carleton.ca.

investigating the asymptotics of the loss probability for an M/G/1/N queue with vacations, will go along this direction.

Now we introduce this system based on the standard M/G/1/N queue. Recall that in the M/G/1/N queue, the arrival process is a Poisson process with rate λ and the service times are i.i.d. random variables with a generic random variable S , which is independent of the arrival process, and has distribution function $S(x)$ and probability density function $s(x)$. The queue has a finite buffer of size $N - 1$ to store the incoming customs. Therefore, N is the maximal system capacity for holding customers including the one in service. To introduce the vacation model M/G/1/N-(V, E)-queue, we further assume in the M/G/1/N queue that whenever the system becomes empty, the server starts a vacation of random length V immediately. If, when the vacation ends, the system is still empty, the server takes another independent vacation of the same length as in the previous one until there is at least one customer waiting in the buffer upon return of the server from a vacation. From then, the server keeps serving customers until the system becomes empty (all customers have been served or exhausted). The vacation time V is assumed to be a non-negative random variable with the distribution function $V(x)$. Then, $\rho = \lambda E[S]$ is the traffic intensity of the model. Throughout the paper, assume that $E[S] < \infty$ and $E[V] < \infty$. Let \mathbb{Z}_+ and \mathbb{C} be the sets of the non-negative integers and the complex numbers, respectively. Denote by a_j and ν_j the probability that j customers join the queue during a service time S and a vacation time V , respectively, i.e., for $j \in \mathbb{Z}_+$,

$$a_j = \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} dS(t),$$

and

$$\nu_j = \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} dV(t).$$

Various performance measures, such as the loss probability, the queue length, the waiting time, the busy period, the system throughput and so on, were investigated for the M/G/1/N-(V, E)-queue in the literature, e.g. [13, 11, 18, 9]. The loss probability, which was studied in [13, 11, 9], has been treated as a central issue among these performance measures. In these works, the loss probability for the finite M/G/1/N-(V, E)-queue was expressed in terms of the invariant probabilities for the embedded queue of the finite or infinite buffer queues, which are again unknown. As far as we know, for the model studied here no asymptotic analysis, as $N \rightarrow \infty$, for the loss probability has been reported, which stimulated us to provide a detailed characterization of the asymptotic analysis of the loss probability. In the literature, asymptotic results for the M/G/1/N queue (without server vacations) can be found, for example, in [3]. In addition, asymptotic rates for the loss probability of the GI/M/1/N queue were obtained in ([3, 6]), by using the duality relationship between the M/G/1/N queue and the GI/M/1/N queue. However, asymptotic analysis of the loss probability seems not complete even for the M/G/1/N and GI/M/1/N queues and some important cases remain unexplored, which stimulated us to introduce new asymptotic methods to study the loss probability for the more general M/G/1/N-(V, E)-queue so that the methods or the results can be applied to derive new asymptotic behavior for those queues.

In this paper we will provide a detailed characterization of asymptotic behavior of the loss probability for the M/G/1/N-(V,E)-queue. Some of the results are intuitively expected and others are not, and some of results are easy consequences from the existing literature and others deserve more detailed analysis. The characterization is obtained by investigating the dominant singular point in terms of singularity analysis, such as the Tauberian theorem, Tauberian-like

theorem (Theorem A.1 in Appendix) or a direct complex analysis. As special cases, for the standard M/G/1/N queue and the GI/M/1/N queue, we will provide new asymptotic results. Specifically, our main contributions include: (1) For $\rho < 1$ and under a light-tailed condition, two cases are studied in Theorem 3.1: for the first case, the decay is light but not exactly geometric, a new asymptotic behavior, and for the second case, decay is exactly geometric; (2) Also for $\rho < 1$, asymptotic results when either the service or the vacation variable is heavy-tailed are provided in Theorem 3.2. This case was not considered in [3]; (3) For $\rho = 1$, asymptotics are studied in Theorem 3.3 under the assumption that the second moment of the service time is finite and infinite, respectively. The infinite case was not considered in [3]; (4) An asymptotic result for $\rho > 1$ is reported in Theorem 3.4, which can be expected from the corresponding result in [3]; (5) When $V = 0$, the vacation model is degenerated to the standard M/G/1/N queue, for which asymptotic results are summarized in Theorem 4.1. We also provide asymptotic results for the GI/M/1/N queue in Theorem 4.2 by using the dual relationship with the M/G/1/N queue. These two theorems provide extensions of existing literature results.

The rest of the paper is organized into: Section 2, which bridges the asymptotics of the loss probability of the finite queue and decay behavior of the invariant measure of the corresponding infinite queue; Section 3, which is devoted to asymptotic analysis of the M/G/1/N-(V,E)-queue and contains main results of the paper; Section 4, in which asymptotic results for the standard M/G/1/N queue the GI/M/1/N queue are provided; Section 5, which provides concluding remarks; and an Appendix, which contains some properties in asymptotic analysis used in this paper for convenience.

2 Connection with the M/G/1/ ∞ -(V, E)-queue

Let $L(t)$ be the queue length of the M/G/1/N-(V,E)-queue at time t . Since the state space is finite, the system $L(t)$ is always stable. It is known that $L(t)$ itself is not a Markov process, but it can be analyzed through its embedded chain $L(\tau_n)$ at the departure epochs τ_n , $n \in \mathbb{Z}_+$. The transition matrix of the embedded chain $L(\tau_n)$ is given by

$$P(N) = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{N-2} & 1 - \sum_{k=0}^{N-2} b_k \\ a_0 & a_1 & a_2 & \dots & a_{N-2} & 1 - \sum_{k=0}^{N-2} a_k \\ 0 & a_0 & a_1 & \dots & a_{N-3} & 1 - \sum_{k=0}^{N-3} a_k \\ 0 & 0 & a_0 & \dots & a_{N-4} & 1 - \sum_{k=0}^{N-4} a_k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_0 & 1 - a_0 \end{pmatrix}_{N \times N},$$

where $b_j = \sum_{i=1}^{j+1} \frac{\nu_i}{1-\nu_0} a_{j-i+1}$, $j \in \mathbb{Z}_+$. Obviously, $P(N)$ is positive recurrent. Let $\pi(N)$ be the probability invariant vector of $P(N)$, i.e. $\pi(N)P(N) = \pi(N)$, or

$$\pi_j(N) = \pi_0(N)b_j + \sum_{k=1}^{j+1} \pi_k(N)a_{j+1-k}, \quad j = 0, \dots, N-2, \quad (2.1)$$

and $\sum_{k=0}^{N-1} \pi_k(N) = 1$.

Let $\pi_j^*(N) = \lim_{t \rightarrow \infty} P[L(t) = j]$, $j = 0, 1, \dots, N$. By the Poisson Arrival See Time Average Property, we see that $\pi_j^*(N)$ is also the probability that there are j customers in the system just

before an arrival, which implies that $P_{\text{loss}}(N) = \pi_N^*(N)$. By Theorem 4.2 in [9], we have

$$P_{\text{loss}}(N) = 1 - \frac{(1 - \nu_0)\lambda^{-1}}{E[V]\pi_0(N) + E[S](1 - \nu_0)}. \quad (2.2)$$

Therefore, the asymptotic analysis of the loss probability solely depends on the property of $\pi_0(N)$. In the following we establish a relationship between the stationary probabilities for the finite-buffer queue and the invariant measure for the corresponding infinite-buffer queue for all values of the traffic intensity ρ .

To connect the loss probability $P_{\text{loss}}(N)$ with the invariant measure of the corresponding infinite queue, we now introduce some related notations. For sequences $\{x_n, n \in \mathbb{Z}_+\}$ and $\{y_n, n \in \mathbb{Z}_+\}$ of non-negative real numbers, define

$$S_x(n) = \sum_{j=0}^{n-1} x_j, \quad \bar{S}_x(n) = \sum_{j=n}^{\infty} x_j, \quad n \geq 1,$$

and write $x_n \sim y_n$ to imply $\lim_{n \rightarrow \infty} x_n/y_n = 1$. Similarly, for two functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ as $x \rightarrow x_0 -$ stands for $\lim_{x \rightarrow x_0 -} f(x)/g(x) = 1$, and $f(x) \sim g(x)$ for $\lim_{x \rightarrow +\infty} f(x)/g(x) = 1$. For any distribution function $F(x)$ on the nonnegative real line, let $f(x)$ be its probability density function when it exists, $\bar{F}(x) = 1 - F(x)$ be its complementary distribution function,

$$F^*(t) = \int_0^{\infty} e^{-tx} dF(x) = \int_0^{\infty} e^{-tx} f(x) dx, \quad t \in \mathbb{C}$$

be its Laplace transform, and

$$F_e(x) = \frac{1}{E[X]} \int_0^x \bar{F}(u) du$$

be its equilibrium distribution whenever $E[X]$ is finite. The distribution function $F(x)$ is said to be regularly varying if $\bar{F}(x) = x^{-\alpha} L(x)$, where $L(x)$ is a slowly varying function that satisfies $\lim_{x \rightarrow +\infty} \frac{L(tx)}{L(x)} = 1$ for any $t > 0$.

The corresponding infinite-buffer vacation model is referred to as the M/G/1/ ∞ -(V, E)-queue, and its embedded chain has the following transition matrix:

$$P = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where the coefficients b_i and a_i are the same as those in $P(N)$. Regardless of stability or non-stability of the infinite-buffer queue, P has a unique invariant measure, up to a multiplicative constant, due to its special structure. Let π be the invariant measure of P , i.e. $\pi P = \pi$, or

$$\pi_j = \pi_0 b_j + \sum_{k=1}^{j+1} \pi_k a_{j+1-k}, \quad j \geq 0. \quad (2.3)$$

Define the generating functions

$$B(z) = \sum_{k=0}^{\infty} b_k z^k, \quad A(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \Pi(z) = \sum_{k=0}^{\infty} \pi_k z^k$$

for b_k , a_k and π_k , respectively. It is routine to calculate

$$A(z) = S^*(\lambda - \lambda z), \quad B(z) = \frac{S^*(\lambda - \lambda z)[V^*(\lambda - \lambda z) - \nu_0]}{z(1 - \nu_0)}.$$

From (2.3), we have

$$\Pi(z)[z - A(z)] = \pi_0[zB(z) - A(z)],$$

whenever $\Pi(z)$ is finite, that is,

$$\Pi(z)[z - S^*(\lambda - \lambda z)] = \frac{\pi_0[V^*(\lambda - \lambda z) - 1]S^*(\lambda - \lambda z)}{1 - \nu_0}. \quad (2.4)$$

Observing that all equations of (2.1) are identical to the corresponding ones in (2.3), we have

$$\pi_i(N) = \frac{\pi_i}{S_\pi(N)}, \quad i = 0, 1, \dots, N - 1. \quad (2.5)$$

Remark 2.1. Note that P is positive recurrent, or π is a probability measure, if and only if $\rho < 1$.

The following lemma will later be frequently used to analyze the asymptotic behavior of the loss probability in Section 3.

Lemma 2.1. For the case of $\rho < 1$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{\bar{S}_\pi(N)} P_{\text{loss}}(N) = 1 - \rho; \quad (2.6)$$

and for the case of $\rho \geq 1$, we have

$$\lim_{N \rightarrow \infty} S_\pi(N) \left[P_{\text{loss}}(N) - \left(1 - \frac{1}{\rho} \right) \right] = \frac{E[V]\pi_0}{\rho E[S](1 - \nu_0)}. \quad (2.7)$$

Proof. Inserting (2.4) into (2.2) yields

$$P_{\text{loss}}(N) = \frac{E[V]\pi_0 + (E[S] - \lambda^{-1})(1 - \nu_0)S_\pi(N)}{E[V]\pi_0 + E[S](1 - \nu_0)S_\pi(N)}. \quad (2.8)$$

When $\rho < 1$, we have $\sum_{i=0}^{\infty} \pi_i = 1$ and $\pi_0 = \frac{(1-\rho)(1-\nu_0)}{\lambda E[V]}$. Use (2.8) to derive

$$P_{\text{loss}}(N) = \frac{(1 - \rho)\bar{S}_\pi(N)}{(1 - \rho) + \rho S_\pi(N)},$$

from which (2.6) follows easily.

When $\rho \geq 1$, we have from (2.8)

$$\lim_{N \rightarrow \infty} P_{\text{loss}}(N) = 1 - \frac{1}{\rho},$$

since $S_\pi(N) \rightarrow \infty$ as $N \rightarrow \infty$. Write (2.8) as

$$P_{\text{loss}}(N) - \left(1 - \frac{1}{\rho} \right) = \frac{E[V]\pi_0}{\rho E[V]\pi_0 + \rho S_\pi(N)E[S](1 - \nu_0)}.$$

Then, (2.7) follows immediately. \square

From Lemma 2.1, we see that the asymptotic behavior of the loss probability $P_{\text{loss}}(N)$ as $N \rightarrow \infty$ relies on the decay property in the invariant measure π . For a stable system, we study the decay rate of the tail probability $\overline{S}_\pi(N)$, and for a non-stable system, we analyze the decay of $1/S_\pi(N)$.

For the Laplace transform $F^*(\lambda - \lambda z)$ ($F^*(\lambda - \lambda z) = F^*(t)|_{t=\lambda-\lambda z}$), define $R_{F^*} \geq 1$ to be the leftmost singular point of the functions $F^*(\lambda - \lambda z)$,

$$F^{*'}(\lambda - \lambda z) = \frac{dF^*(\lambda - \lambda u)}{du} \Big|_{u=z} = \int_0^\infty \lambda t e^{-(\lambda-\lambda z)t} f(t) dt,$$

and

$$F^{*''}(\lambda - \lambda z) = \frac{d^2 F^*(\lambda - \lambda u)}{du^2} \Big|_{u=z} = \int_0^\infty (\lambda t)^2 e^{-(\lambda-\lambda z)t} f(t) dt.$$

Observe that $S^*(\lambda - \lambda \cdot 1) = 1$, $S^{*'}(\lambda - \lambda \cdot 1) = \rho$ and $S^*(\lambda - \lambda z)$ is a strictly increasing and convex function about z when $z \geq 0$. This observation yields the following lemma, which will later be used to investigate the decay of $\overline{S}_\pi(N)$ and $1/S_\pi(N)$.

Lemma 2.2. (i) If $\rho < 1$ and $1 < R_{S^*} < S^*(\lambda - \lambda R_{S^*})$, then there exists at most one solution z_1 to the equation $z = S^*(\lambda - \lambda z)$, $1 < z < R_{S^*}$.

(ii) If $\rho > 1$, then there exists exactly one solution z_2 to the equation $z = S^*(\lambda - \lambda z)$, $0 < z < 1$.

3 Asymptotic behavior for the M/G/1/N-(V, E)-queue

We perform the asymptotic analysis according to three cases: (i) $\rho < 1$; (ii) $\rho = 1$; and (iii) $\rho > 1$.

3.1 The case when $\rho < 1$

For this case, we investigate the asymptotic behavior for two subcases according to $\min\{R_{S^*}, R_{V^*}\} > 1$ or $\min\{R_{S^*}, R_{V^*}\} = 1$. The former subcase corresponds to light-tailed characterizations, for which the asymptotic rates are either exactly geometric or not exactly geometric. The latter subcase corresponds to that one of the sequences $\{b_i\}$ and $\{a_i\}$ is heavy-tailed, for which a regular varying condition is imposed on the service time and vacation time distributions.

Theorem 3.1. Suppose that $\rho < 1$ and $r := \min\{R_{S^*}, R_{V^*}\} > 1$.

(i) Suppose that $r = R_{V^*}$ and $S^*(\lambda - \lambda r) < r$. If $V^*(\lambda - \lambda z)$ can be analytically extended to a $\Delta(r)$ -domain (see Definition A.1 in Appendix) and for some $\theta > 0$, $V^*(\lambda - \lambda z) \sim \frac{c}{(r-z)^\theta}$, as $z \rightarrow r, z \in \Delta(r)$, then

$$\lim_{N \rightarrow \infty} N^{1-\theta} r^N P_{\text{loss}}(N) = \frac{c(1-\rho)^2 S^*(\lambda - \lambda r)}{r^{\theta-1} \Gamma(\theta) \lambda E[V] (r-1) [r - S^*(\lambda - \lambda r)]}. \quad (3.1)$$

(ii) Suppose that $r = R_{s^*}$ and $r < S^*(\lambda - \lambda r)$, then there exists exactly one solution z_1 to the equation $z = S^*(\lambda - \lambda z)$, $1 < z < r$, such that

$$\lim_{N \rightarrow \infty} z_1^N P_{loss}(N) = \frac{z_1(1-\rho)^2[1-V^*(\lambda-\lambda z_1)]}{\lambda E[V](z_1-1)[1-S^{*'}(\lambda-\lambda z_1)]}. \quad (3.2)$$

Proof. (i) Due to the assumption, we know that $V^*(\lambda - \lambda z)$ is analytic in a $\Delta(r)$ -domain, which implies that there exists some r_1 , $r_1 > r$ and some α , $0 < \alpha < \frac{\pi}{2}$ such that $V^*(\lambda - \lambda z)$ is analytic in the domain $\Delta(r, \alpha, r_1)$. Thus we can find an r_2 such that $r < r_2 < r_1$ and $V^*(\lambda - \lambda z)$ is analytic in the domain $\Delta(r, \alpha, r_2)$. Hence by the expression (2.4), we know that $\Pi(z)$ can be analytically extended to this domain $\Delta(r, \alpha, r_2)$, i.e. $\Pi(z)$ is $\Delta(r)$ -analytic.

Since $V^*(\lambda - \lambda z) \sim \frac{c}{(r-z)^\theta}$ as $z \rightarrow r$ in the complex domain $\Delta(r, \alpha, r_2)$, we have

$$\lim_{z \rightarrow r} \Pi(z)(r-z)^\theta = \frac{c(1-\rho)S^*(\lambda-\lambda r)}{\lambda E[V][r-S^*(\lambda-\lambda r)]},$$

that is,

$$\sum_{n=0}^{\infty} \pi_n r^n \left(\frac{z}{r}\right)^n \sim \frac{c(1-\rho)S^*(\lambda-\lambda r)}{\lambda E[V][r-S^*(\lambda-\lambda r)]} \frac{1}{r^\theta(1-\frac{z}{r})^\theta}, \quad \text{as } z \rightarrow r, z \in \Delta(r, \alpha, r_2).$$

It follows essentially from a result in [8] (see Theorem A.1 in Appendix) that

$$\pi_n \sim \frac{c(1-\rho)S^*(\lambda-\lambda r)}{r^\theta \Gamma(\theta) \lambda E[V][r-S^*(\lambda-\lambda r)]} n^{\theta-1} r^{-n},$$

from which and the Stolz-Cesàro theorem, we have

$$\overline{S}_\pi(n) \sim \frac{c(1-\rho)S^*(\lambda-\lambda r)}{r^{\theta-1} \Gamma(\theta) \lambda E[V](r-1)[r-S^*(\lambda-\lambda r)]} n^{\theta-1} r^{-n}.$$

Then, assertion (3.1) follows from Lemma 2.1 immediately.

(ii) It follows from Lemma 2.2 that there exists exactly one solution z_1 to the equation $z = S^*(\lambda - \lambda z)$, $1 < z < r$. Since

$$\lim_{z \rightarrow z_1^-} \Pi(z)(z_1 - z) = \frac{z_1(1-\rho)[1-V^*(\lambda-\lambda z_1)]}{\lambda E[V][1-S^{*'}(\lambda-\lambda z_1)]},$$

z_1 is a simple pole for $\Pi(z)$. For any $z \in \mathbb{C}$ such that $|z| = z_1$ and $z \neq z_1$, we have

$$|S^*(\lambda - \lambda z)| \leq \int_0^\infty e^{-\lambda x} |e^{\lambda z x}| dS(x) < S^*(\lambda - \lambda |z|) = z_1,$$

which implies that there is only one solution $z = z_1$ to the equation $z = S^*(\lambda - \lambda z)$ on the circle $|z| = z_1$. Hence $z = z_1$ is the only singular point of $\Pi(z)$ on the circle $|z| = z_1$. By Theorem 5.2.1 in [20], a standard result on asymptotics of complex functions, we have

$$\pi_n \sim \frac{(1-\rho)[1-V^*(\lambda-\lambda z_1)]}{\lambda E[V][1-S^{*'}(\lambda-\lambda z_1)]} z_1^{-n},$$

from which and the Stolz-Cesàro theorem, it follows that

$$\overline{S}_\pi(n) \sim \frac{z_1(1-\rho)[1-V^*(\lambda-\lambda z_1)]}{\lambda E[V](z_1-1)[1-S^{*'}(\lambda-\lambda z_1)]} z_1^{-n}.$$

The assertion (3.2) follows from Lemma 2.1 immediately. \square

Remark 3.1. (i) In the above theorem, the assumption $S^*(\lambda - \lambda r) < r$ in (i), which is equivalent to that there is no solution to the equation $z = S^*(\lambda - \lambda z)$ for $1 < z < r$, ensures that the dominant singular point of $\Pi(z)$ is determined by $V^*(\lambda - \lambda z)$, while the assumption in (ii) ensures that the dominant singular point of $\Pi(z)$ is determined by the solution to the equation $z = S^*(\lambda - \lambda z)$ for $1 < z < r$. (ii) In the first case, the exact asymptotic behavior of $P_{loss}(N)$ reveals a non-geometric phenomenon when $\theta \neq 1$, which is faster than the geometric rate r^{-N} when $\theta < 1$ and slower than the geometric rate r^{-N} when $\theta > 1$. This is a different characterization from the well-known exact geometric decay in the related literature.

Example 3.1. We now give an example to illustrate that how to verify the analytic condition in (i) of Theorem 3.1. Assume that (i): $\lambda = 1$, V has probability density function $v(x) = \frac{\alpha^{k+1} x^k}{k!} e^{-\alpha x}$, $\alpha > 0$, $x \geq 0$, and S has probability density function $s(x) = p e^{-p x}$, $p > 0$, $x \geq 0$. Then it is easy to calculate

$$E[V] = \frac{k+1}{\alpha}, \quad E[S] = \frac{1}{p}, \quad \rho = \lambda E[S] = 1/p,$$

$$V^*(\lambda - \lambda z) = V^*(1 - z) = \int_0^\infty e^{-(1-z)x} v(x) dx = \frac{\alpha^{k+1}}{(1 + \alpha - z)^{k+1}}, \quad \text{Re}(z) < 1 + \alpha,$$

and

$$S^*(\lambda - \lambda z) = S^*(1 - z) = \int_0^\infty e^{-(1-z)x} s(x) dx = \frac{p}{1 + p - z}, \quad \text{Re}(z) < 1 + p.$$

So, we have $R_{V^*} = 1 + \alpha$, $R_{S^*} = 1 + p$. Assume that (ii): $p > 1 + \alpha$, then we have $r = R_{V^*}$ and $S^*(\lambda - \lambda r) < r$. Obviously, the function $\frac{\alpha^{k+1}}{(1 + \alpha - z)^{k+1}}$ is analytic in $\mathbb{C} \setminus \{1 + \alpha\}$, i.e. the whole complex plane except for the point $z = 1 + \alpha$. Hence, the Laplace transform $V^*(\lambda - \lambda z)$ can be analytically extended to $\mathbb{C} \setminus \{1 + \alpha\}$, and the $\Delta(r)$ -analytic condition holds automatically. Thus, under the assumptions (i) and (ii), every condition in (i) of Theorem 3.1 is satisfied, so we have

$$\lim_{N \rightarrow \infty} N^{-k} (1 + \alpha)^N P_{loss}(N) = \frac{\alpha^k p (1 - 1/p)^2}{(1 + \alpha)^k (k + 1)! (p - 1 - \alpha)}.$$

□

Theorem 3.2. Assume that $\rho < 1$ and $\min\{R_{S^*}, R_{V^*}\} = 1$.

(i) If $\overline{V}(x) \sim c \overline{S}(x) \sim x^{-\alpha} L(x)$ for some $c > 0$ and $\alpha > 1$, then

$$\lim_{N \rightarrow \infty} \frac{N^{\alpha-1}}{L(\frac{N}{\lambda})} P_{loss}(N) = \left(\frac{1 - \rho}{E[V]} + \frac{\rho}{c E[S]} \right) \frac{\lambda^{\alpha-1}}{\alpha - 1}. \quad (3.3)$$

(ii) If $\overline{V}(x) = o(\overline{S}(x))$ and $\overline{S}(x) \sim x^{-\alpha} L(x)$ for some $\alpha > 1$, then

$$\lim_{N \rightarrow \infty} \frac{N^{\alpha-1}}{L(\frac{N}{\lambda})} P_{loss}(N) = \frac{\lambda^{\alpha}}{\alpha - 1}. \quad (3.4)$$

(iii) If $\overline{S}(x) = o(\overline{V}(x))$ and $\overline{V}(x) \sim x^{-\alpha} L(x)$ for some $\alpha > 1$, then

$$\lim_{N \rightarrow \infty} \frac{N^{\alpha-1}}{L(\frac{N}{\lambda})} P_{loss}(N) = \frac{(1 - \rho) \lambda^{\alpha-1}}{(\alpha - 1) E[V]}. \quad (3.5)$$

Proof. (i) It was shown in Proposition 1.5.10 in [5] that if $\bar{F}(x) \sim x^{-\alpha}L(x)$ for some $\alpha > 1$, then the equilibrium distribution

$$\bar{F}_e(x) \sim \frac{x^{-(\alpha-1)}L(x)}{E[X](\alpha-1)}. \quad (3.6)$$

Using L'Hospital rule obtains

$$\lim_{x \rightarrow \infty} \frac{\bar{V}_e(x)}{\bar{S}_e(x)} = \lim_{x \rightarrow \infty} \frac{E[S](E[V] - \int_0^x \bar{V}(t)dt)}{E[V](E[S] - \int_0^x \bar{S}(t)dt)} = \frac{E[S]}{E[V]} \lim_{x \rightarrow \infty} \frac{\bar{V}(x)}{\bar{S}(x)}, \quad (3.7)$$

whenever the limit $\lim_{x \rightarrow \infty} \frac{\bar{V}(x)}{\bar{S}(x)}$ exists. Using (3.6) and (3.7) yields

$$\bar{V}_e(x) \sim \frac{cE[S]}{E[V]} \bar{S}_e(x) \sim \frac{x^{-(\alpha-1)}L(x)}{E[V](\alpha-1)}.$$

Then by Proposition 6.4 in [2], we have

$$\bar{S}_\pi(k) \sim \left(\frac{1}{E[V]} + \frac{\rho}{c(1-\rho)E[S]} \right) \frac{\lambda^{\alpha-1}}{\alpha-1} k^{-(\alpha-1)} L\left(\frac{k}{\lambda}\right),$$

from which and Lemma 2.1, (3.3) follows.

(ii) It follows from (3.6) and (3.7) that

$$\bar{S}_e(x) \sim \frac{x^{-(\alpha-1)}L(x)}{E[S](\alpha-1)}, \quad \bar{V}_e(x) = o(\bar{S}_e(x)).$$

From Proposition 6.1 in [2], we derive

$$\bar{S}_\pi(k) \sim \frac{\lambda^\alpha}{(1-\rho)(\alpha-1)} k^{-(\alpha-1)} L\left(\frac{k}{\lambda}\right),$$

from which and Lemma 2.1, (3.4) follows.

(iii) It follows from (3.6) and (3.7) that

$$\bar{V}_e(x) \sim \frac{x^{-(\alpha-1)}L(x)}{E[V](\alpha-1)}, \quad \bar{S}_e(x) = o(\bar{V}_e(x)).$$

From Proposition 6.2 in [2], we derive

$$\bar{S}_\pi(k) \sim \frac{\lambda^{\alpha-1}}{(\alpha-1)E[V]} k^{-(\alpha-1)} L\left(\frac{k}{\lambda}\right),$$

from which and Lemma 2.1, (3.5) follows. \square

Remark 3.2. (i) Under the assumption that $\bar{S}(x) \sim x^{-\alpha}L(x)$ ($\bar{V}(x) \sim x^{-\alpha}L(x)$), it is natural to restrict that $\alpha > 1$ due to $E[S] < \infty$ ($E[V] < \infty$), the basic assumptions of this model. (ii) Heavy-tailed asymptotic results were obtained for M/G/1-type Markov chains (e.g. [15, 19]), in terms of asymptotics of $a(n)$ and $b(n)$, which can not trivially lead to asymptotic results in terms of queueing primitives or service/arrival parameters, for example, results presented in this paper.

3.2 The case when $\rho = 1$

For this case, if the second moment of the service time is finite, we provide the asymptotic decay for the loss probability. When the second moment is infinite, we consider a family of service distributions to understand the asymptotic behavior.

Theorem 3.3. *Assume that $\rho = 1$.*

(i) *If $E[S^2] < \infty$, then*

$$\lim_{N \rightarrow \infty} NP_{loss}(N) = \frac{\lambda^2 E[S^2]}{2}. \quad (3.8)$$

(ii) *If $E[S^2] = \infty$, let $s(x) \sim cx^{-\theta}$ for $2 < \theta \leq 3$, where c is a constant. Then, for $2 < \theta < 3$,*

$$\lim_{N \rightarrow \infty} N^{\theta-2} P_{loss}(N) = \frac{c\lambda^{\theta-1}\Gamma(\theta-1)\Gamma(4-\theta)}{(1-\theta)(2-\theta)(3-\theta)}, \quad (3.9)$$

and for $\theta = 3$,

$$\lim_{N \rightarrow \infty} \frac{N}{\ln N} P_{loss}(N) = \frac{c\lambda^2}{2}. \quad (3.10)$$

Proof. (i) Since

$$\lim_{z \rightarrow 1-} \frac{z - S^*(\lambda - \lambda z)}{(1-z)^2} = \lim_{z \rightarrow 1-} \frac{1 - S^{*'}(\lambda - \lambda z)}{-2(1-z)} = \frac{-S^{*''}(\lambda - \lambda \cdot 1)}{2} = -\frac{\lambda^2 E[S^2]}{2},$$

we have

$$\lim_{z \rightarrow 1-} \Pi(z)(1-z) = \frac{-2\pi_0}{(1-\nu_0)\lambda^2 E[S^2]} \lim_{z \rightarrow 1-} \frac{V^*(\lambda - \lambda z) - 1}{1-z} = \frac{2\pi_0 \lambda E[V]}{(1-\nu_0)\lambda^2 E[S^2]}.$$

It follows from a Tauberian theorem (Theorem A.3 in the Appendix) that

$$S_\pi(n) \sim \frac{2\pi_0 \lambda E[V]n}{(1-\nu_0)\lambda^2 E[S^2]}.$$

From Lemma 2.1, we have

$$\lim_{N \rightarrow \infty} NP_{loss}(N) = \frac{\lambda E[S^2]}{2\rho E[S]},$$

which is (3.8) by noting that $\rho = \lambda E[S] = 1$.

(ii) Let $U(x) = \int_0^x (\lambda t)^2 s(t) dt$. There are two cases.

(1) If $2 < \theta < 3$, then we have

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x^{3-\theta}} = \frac{c\lambda^2}{3-\theta}.$$

Obviously,

$$S^{*''}(\lambda - \lambda z) = \int_0^\infty (\lambda t)^2 e^{-(\lambda - \lambda z)t} s(t) dt = \int_0^\infty e^{-(\lambda - \lambda z)t} dU(t).$$

Using a Tauberian theorem (Theorem A.2 in the Appendix) yields

$$S^{*''}(\lambda - \lambda z) \sim \frac{c\lambda^{\theta-1}\Gamma(4-\theta)}{3-\theta}(1-z)^{-(3-\theta)}, \quad z \rightarrow 1-.$$

By L'Hospital rule, we have

$$\lim_{z \rightarrow 1-} \frac{z - S^*(\lambda - \lambda z)}{(1-z)^{-(1-\theta)}} = \lim_{z \rightarrow 1-} \frac{1 - S^{*'}(\lambda - \lambda z)}{-(1-\theta)(1-z)^{-(2-\theta)}} = \frac{-c\lambda^{\theta-1}\Gamma(4-\theta)}{(1-\theta)(2-\theta)(3-\theta)},$$

from which it follows that

$$\begin{aligned} \lim_{z \rightarrow 1-} \Pi(z)(1-z)^{-(2-\theta)} &= \frac{\pi_0}{(1-\nu_0)} \lim_{z \rightarrow 1-} \frac{[V^*(\lambda - \lambda z) - 1](1-z)^{-(2-\theta)}}{\frac{-c\lambda^{\theta-1}\Gamma(4-\theta)}{(1-\theta)(2-\theta)(3-\theta)}(1-z)^{-(1-\theta)}} \\ &= \frac{\pi_0(1-\theta)(2-\theta)(3-\theta)\lambda E[V]}{c(1-\nu_0)\lambda^{\theta-1}\Gamma(4-\theta)}. \end{aligned}$$

Now, by using another Tauberian theorem (Theorem A.3), we have

$$S_\pi(n) \sim \frac{\pi_0(1-\theta)(2-\theta)(3-\theta)\lambda E[V]}{c(1-\nu_0)\lambda^{\theta-1}\Gamma(4-\theta)} \frac{n^{\theta-2}}{\Gamma(\theta-1)},$$

from which and Lemma 2.1, (3.9) follows immediately.

(2) If $\theta = 3$, then

$$\lim_{x \rightarrow \infty} \frac{U(x)}{\ln x} = c\lambda^2.$$

Using a Tauberian theorem (Theorem A.2) yields

$$S^{*''}(\lambda - \lambda z) \sim c\lambda^2 \ln \left(\frac{1}{\lambda - \lambda z} \right) = -c\lambda^2 \ln(\lambda - \lambda z).$$

Then, by L'Hospital rule, we obtain

$$\lim_{z \rightarrow 1-} \frac{V^*(\lambda - \lambda z) - 1}{z - 1} = \lambda E[V], \quad (3.11)$$

and

$$\begin{aligned} \lim_{z \rightarrow 1-} \frac{z - S^*(\lambda - \lambda z)}{(1-z)^2 \ln(\lambda - \lambda z)} &= \lim_{z \rightarrow 1-} \frac{1 - S^{*'}(\lambda - \lambda z)}{(1-z)[2 \ln(\lambda - \lambda z) - (1-z)]} \\ &= \lim_{z \rightarrow 1-} \frac{-S^{*''}(\lambda - \lambda z)}{2 \ln(\lambda - \lambda z) + 2(1-z) + 2} \\ &= \frac{c\lambda^2}{2}. \end{aligned} \quad (3.12)$$

It follows from (3.11) and (3.12) that

$$\begin{aligned} \lim_{z \rightarrow 1-} \Pi(z)(1-z) \ln \left(\frac{1}{\lambda - \lambda z} \right) &= \frac{\pi_0}{(1-\nu_0)} \lim_{z \rightarrow 1-} \frac{[V^*(\lambda - \lambda z) - 1]S^*(\lambda - \lambda z)}{z - S^*(\lambda - \lambda z)} (1-z) \ln \left(\frac{1}{\lambda - \lambda z} \right) \\ &= \frac{2\pi_0 E[V]}{c(1-\nu_0)\lambda}. \end{aligned}$$

Hence, we have

$$\Pi(z) \sim \frac{1}{(1-z)\ln(\frac{1}{\lambda-\lambda z})} \sim \frac{2\pi_0 E[V]}{c(1-\nu_0)\lambda} \frac{1}{(1-z)\ln(\frac{1}{1-z})}, \quad \text{as } z \rightarrow 1-.$$

Using another Tauberian theorem (Theorem A.3) leads to

$$S_\pi(n) \sim \frac{2\pi_0 E[V]}{c(1-\nu_0)\lambda} \frac{n}{\ln n},$$

from which and Lemma 2.1, (3.9) follows immediately. \square

Remark 3.3. (a) In the assumption of $s(x) \sim cx^{-\theta}$ in case (ii), it is reasonable to restrict θ in the domain $(2, 3]$. Otherwise, if $\theta = 2$, then $E[S] = \infty$, and if $\theta > 3$, then $E[S^2] < \infty$, both contradict other assumptions. (b) The condition that $E[S^2] < \infty$ is often imposed in studies of this type of problems, for example in [3], but as far as we know, Case (ii) is a new one. (c) In the case $\rho = 1$, the asymptotic behavior of the loss probability is independent of the vacation time V .

3.3 The case when $\rho > 1$

This is the simplest case, in which we no longer need to impose any further property assumptions on the service time or on the vacation time for the following asymptotic property for the loss probability.

Theorem 3.4. Assume that $\rho > 1$. Then we have

$$\lim_{N \rightarrow \infty} z_2^{-N} \left[P_{\text{loss}}(N) - \left(1 - \frac{1}{\rho} \right) \right] = \frac{(1-z_2)[1-S^{*'}(\lambda-\lambda z_2)]E[V]}{z_2 \rho E[S][1-V^*(\lambda-\lambda z_2)]}, \quad (3.13)$$

where $z_2 \in (0, 1)$ is the unique solution to the equation $z = S^*(\lambda - \lambda z)$.

Proof. From Lemma 2.2, we know that there exists a unique solution $z_2 \in (0, 1)$ to the equation $z = S^*(\lambda - \lambda z)$. Obviously, $S^{*'}(\lambda - \lambda z_2) < 1$. Since

$$\lim_{z \rightarrow z_2-} \Pi(z)(z_2 - z) = \frac{z_2 \pi_0 [1 - V^*(\lambda - \lambda z_2)]}{(1 - \nu_0) [1 - S^{*'}(\lambda - \lambda z_2)]},$$

z_2 is a simple pole for $\Pi(z)$. Similar to the proof of Theorem 3.1, we know that $z = z_2$ is the unique singular point of $\Pi(z)$ on the circle $|z| = z_2$. By Theorem 5.2.1 in [20], a standard result on asymptotics of complex functions, we have

$$\pi_n \sim \frac{\pi_0 [1 - V^*(\lambda - \lambda z_2)]}{(1 - \nu_0) [1 - S^{*'}(\lambda - \lambda z_2)]} z_2^{-n},$$

from which and the Stolz-Cesàro theorem it follows that

$$S_\pi(n) \sim \frac{z_2 \pi_0 [1 - V^*(\lambda - \lambda z_2)]}{(1 - z_2)(1 - \nu_0) [1 - S^{*'}(\lambda - \lambda z_2)]} z_2^{-n}.$$

The assertion (3.13) follows now from Lemma 2.1 immediately. \square

4 Decay for standard M/G/1/N and GI/M/1/N queues

The M/G/1/N queue with vacations becomes the (standard) M/G/1/N queue when the vacation time $V = 0$. The embedded transition matrix of the M/G/1/N queue is obtained by replacing b_k by a_k , $k = 0, \dots, N-2$, in the first row of the embedded transition matrix for the M/G/1/N queue with vacations. Denote by $\hat{P}_{\text{loss}}(K)$ the loss probability for the M/G/1/N queue. It is known that

$$\hat{P}_{\text{loss}}(N) = 1 - \frac{1}{\hat{\pi}_0(N) + \rho},$$

where $\hat{\pi}(N)$ is the invariant vector of the embedded Markov chain for the M/G/1/N queue. The following result is obtained by the same asymptotic analysis for $P_{\text{loss}}(N)$ in Section 2 with $V = 0$.

Theorem 4.1. *For the M/G/1/N queue, we have the following asymptotic results on the loss probability $\hat{P}_{\text{loss}}(N)$.*

(i) *Let $\rho < 1$.*

(1) *If $R_{S^*} = 1$ and $\bar{S}(x) \sim x^{-\alpha}L(x)$ for some $\alpha > 1$, then*

$$\lim_{N \rightarrow \infty} \frac{N^{\alpha-1}}{L(\frac{N}{\lambda})} \hat{P}_{\text{loss}}(N) = \frac{\lambda^\alpha}{\alpha - 1}.$$

(2) *If $R_{S^*} > 1$ and $R_{S^*} < S^*(\lambda - \lambda R_{S^*})$, then there exists only one solution σ_1 to the equation $z = S^*(\lambda - \lambda z)$, $1 < z < R_{S^*}$ such that*

$$\lim_{N \rightarrow \infty} \sigma_1^N \hat{P}_{\text{loss}}(N) = \frac{\sigma_1(1 - \rho)^2}{[S^{*'}(\lambda - \lambda \sigma_1) - 1]}.$$

(ii) *Let $\rho = 1$.*

(1) *If $E[S^2] < \infty$, then*

$$\lim_{N \rightarrow \infty} N \hat{P}_{\text{loss}}(N) = \frac{\lambda^2 E[S^2]}{2}.$$

(2) *If $E[S^2] = \infty$ and $s(x) \sim cx^{-\theta}$, then for $2 < \theta < 3$,*

$$\lim_{N \rightarrow \infty} N^{\theta-2} \hat{P}_{\text{loss}}(N) = \frac{c\lambda^{\theta-1}\Gamma(\theta-1)\Gamma(4-\theta)}{(1-\theta)(2-\theta)(3-\theta)},$$

and for $\theta = 3$,

$$\lim_{N \rightarrow \infty} \frac{N}{\ln N} \hat{P}_{\text{loss}}(N) = \frac{c\lambda^2}{2}. \quad (4.1)$$

(iii) *Let $\rho > 1$. then*

$$\lim_{N \rightarrow \infty} \sigma_2^{-N} \left[\hat{P}_{\text{loss}}(N) - \left(1 - \frac{1}{\rho}\right) \right] = \frac{1 - S^{*'}(\lambda - \lambda \sigma_2)}{\sigma_2 \rho^2},$$

where $\sigma_2 \in (0, 1)$ is the unique solution to the equation $z = S^(\lambda - \lambda z)$.*

Remark 4.1. (a) The above results for $\rho > 1$ and for $\rho = 1$ with the assumption that $E[S^2] < \infty$ coincide with those in ([3]). (b) When $\rho < 1$, we show that the asymptotic rate can be exactly geometric or polynomial (heavy-tailed). The later case was neglected in [3]. (c) The asymptotic result for the case that $\rho = 1$ with $E[S^2] = \infty$ is considered new.

We now consider the (standard) $GI/M/1/N$ queue. Denote by A the generic interarrival time random variable, and by $A(x)$ and $a(x)$ its distribution function and probability density function, respectively. Let μ be the service rate and let N be the maximal capacity of the system, or the number of customers in the system including the one in service. The traffic intensity of the $GI/M/1/N$ queues is $\tilde{\rho} = \frac{1}{\mu E[A]}$. By $\tilde{P}_{\text{loss}}(N)$ we denote the loss probability of the $GI/M/1/N$ queue. Using the dual concept (see, e.g. [3, 16]), we have

$$\tilde{P}_{\text{loss}}(N) = \tilde{\pi}_N(N) = \hat{\pi}_0(N+1), \quad (4.2)$$

where $\tilde{\pi}(N)$ is the invariant distribution of the embedded $GI/M/1/N$ queue, and $\hat{\pi}(N+1)$ is the invariant distribution of the $M/G/1/N+1$ queue in which the arrival process is Poissonian with parameter μ , the distribution of the service time $A(x)$ and the traffic intensity $\rho = \tilde{\rho}^{-1}$. From (2.5), (4.2) and the proof of Theorems 3.2–3.3, we obtain the following assertion.

Theorem 4.2. Consider the $GI/M/1/N$ queue.

(i) Let $\tilde{\rho} < 1$. We then have

$$\lim_{N \rightarrow \infty} \eta_1^{-N} \tilde{P}_{\text{loss}}(N) = 1 - A^*(\mu - \mu\eta_1),$$

where $\eta_1 \in (0, 1)$ is the unique solution to the equation $z = A^*(\mu - \mu z)$.

(ii) Let $\tilde{\rho} = 1$.

(1) If $E[A^2] < \infty$, then

$$\lim_{N \rightarrow \infty} N \tilde{P}_{\text{loss}}(N) = \frac{\mu^2 E[A^2]}{2}.$$

(2) If $E[A^2] = \infty$ and $a(x) \sim cx^{-\theta}$, then for $2 < \theta < 3$,

$$\lim_{N \rightarrow \infty} N^{\theta-2} \tilde{P}_{\text{loss}}(N) = \frac{c\mu^{\theta-1}\Gamma(\theta-1)\Gamma(4-\theta)}{(1-\theta)(2-\theta)(3-\theta)},$$

and for $\theta = 3$,

$$\lim_{N \rightarrow \infty} \frac{N}{\ln N} \tilde{P}_{\text{loss}}(N) = \frac{c\mu^2}{2}. \quad (4.3)$$

(iii) Let $\tilde{\rho} > 1$ and let R_{A^*} be the leftmost singular point of the function $A(\mu - \mu z)$.

(1) If $R_{A^*} = 1$ and $\bar{A}(x) \sim x^{-\alpha}L(x)$ for some $\alpha > 1$ as $x \rightarrow \infty$, then

$$\lim_{N \rightarrow \infty} \frac{N^{\alpha-1}}{L(\frac{N}{\lambda})} \left[\tilde{P}_{\text{loss}}(N) - \frac{1}{\tilde{\rho}} \right] = \frac{\lambda^\alpha}{\alpha-1}.$$

(2) If $R_{A^*} > 1$ and $R_{A^*} < S^*(\lambda - \lambda R_{A^*})$, then there exists only one solution η_2 to the equation $z = A^*(\mu - \mu z)$, $z \in (1, R_{A^*})$ such that

$$\lim_{N \rightarrow \infty} \eta_2^N \left[\tilde{P}_{\text{loss}}(N) - \frac{1}{\tilde{\rho}} \right] = \frac{(1-\rho)^2}{A^*(\mu - \mu\eta_2) - 1}.$$

Remark 4.2. (a) The results for $\tilde{\rho} < 1$ and for $\tilde{\rho} = 1$ with $E[A^2] < \infty$ coincide with those in [3, 6]. (b) When $\tilde{\rho} > 1$, we showed that the asymptotic rate can be either geometric or polynomial. The latter case was not considered in ([3]). (c) The asymptotic result on the case that $\tilde{\rho} = 1$ with $E[A^2] = \infty$ is considered new.

5 Conclusions

The main contribution of this paper is a detailed characterization of the asymptotic rates of the loss probability for the M/G/1/N-(V,E)-queue. New asymptotic behavior was found even for simpler M/G/1/N queue and the GI/M/1/N queue in terms of introduction of new asymptotic analysis methods.

Although we only focused on asymptotics of the loss probability, the arguments in this paper can be extended to derive asymptotic results of the other performance measures, such as the queue length distribution. Let $L(t)$ be the queue length process of the M/G/1/ ∞ -(V, E)-queue. Suppose that $L(t)$ is stable, i.e. $\rho < 1$. It is well known that $L(t)$ has the invariant distribution π given by (2.3), i.e. $L(t)$ has the same invariant distribution as that of its embedded queue. Define

$$\|\pi^*(N) - \pi\| = \sum_{j=0}^N |\pi_j^*(N) - \pi_j| + \sum_{j=N+1}^{\infty} \pi_j,$$

where $\pi_N^*(N)$ is given by (2.2) and $\pi_j^*(N)$ is given by (see formula (13) in [9])

$$\pi_j^*(N) = \frac{\pi_j(N)(1 - \nu_0)\lambda^{-1}}{E[V]\pi_0(N) + E[S](1 - \nu_0)}, \quad 0 \leq j \leq N - 1.$$

Since $\pi_j(N) = \frac{\pi_j}{S_\pi(N)}$, $0 \leq j \leq N - 1$, and $\pi_0 = \frac{(1-\rho)(1-\nu_0)}{\lambda E[V]}$, it is not hard to find that the asymptotic rate of $\|\pi^*(N) - \pi\|$ as $N \rightarrow \infty$ is completely determined by the decay rate of $\bar{S}_\pi(N)$ as $N \rightarrow \infty$. Hence the asymptotic rate of $\|\pi^*(N) - \pi\|$ can be derived by using similar arguments to that in Section 3.1.

Acknowledgements

Both authors are deeply grateful to the anonymous referee for very constructive comments, which have allowed the authors to improve the presentation of this paper significantly. This work was supported in part by an NSERC discovery grant of Canada, the Fundamental Research Funds for the Central Universities (grant number 2010QYZD001), and the National Natural Science Foundation of China (grant numbers 10901164, 11071258).

References

- [1] Alfa, A.S. and Zhao, Y.Q. (2000). Overload analysis of the PH/PH/1/K queue and the queue of M/G/1/K type with very large K. *Asia-Pacific J. of OR.* **17**, 123–135.
- [2] Asmussen, S.C. Klüppelberg and Sigman, K. (1999). Sampling at subexponential times, with queueing applications. *Stoch. Proc. Appl.* **79**, 265–286.

- [3] Baiocchi, A. (1992). Asymptotic behaviour of the loss probability of the M/G/1/K and G/M/1/K queues. *Queueing Syst.* **10**, 235–248.
- [4] Baiocchi, A. (1994). Analysis of the loss probability of the MAP/G/1/K queue Part I: asymptotic theory. *Commun. Stat. Stoch. Models* **10**(4), 867–893.
- [5] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. Cambridge University Press.
- [6] Choi, B.D., Kim, B. and Wee, I.-S. (2000). Asymptotic behavior of loss probability in GI/M/1/K queue as K tends to infinity. *Queueing Syst.* **36**, 437–442.
- [7] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed. Wiley, New York.
- [8] Flajolet, P. and Robert, S. (2009). *Analytic Combinatorics*. Cambridge University Press, Cambridge.
- [9] Frey, A. and Takahashi, Y. (1997). A note on an M/G/1/N queue with vacation time and exhaustive service discipline. *Oper. Res. Letters* **21**, 95–100.
- [10] Ishizaki, F. and Takine, T. (1999). Loss probability in a finite discrete-time queue in terms of the steady state distribution of an infinite queue. *Queueing Syst.* **31**, 317–326.
- [11] Keilson, J. and Servi, L.D. (1989). Blocking probability for M/G/1 vacation systems with occupancy level dependent schedules. *Oper. Res.* **37**, 134–140.
- [12] Kim, J. and Kim, B. (2008). Asymptotic analysis for loss probability of queues with finite GI/M/1 type structure. *Queueing Syst.* **57**, 47–55.
- [13] Lee, T.T. (1984). M/G/1/N queue with vacation time and exhaustive service discipline. *Oper. Res.* **32**, 774–784.
- [14] Li, H. and Zhao, Y.Q. (2011). Tail asymptotics for a generalized two-demand queueing model—a kernel method. *Queueing Syst.* **69**, 77–100.
- [15] Li, Q.L. and Zhao, Y.Q. (2005). Heavy-tailed asymptotics of stationary probability vectors of GI/G/1 type. *Adv. Appl. Prob.* **37**, 482–509.
- [16] Miyazawa, M. (1990). Complementary generating functions for the $M^X/GI/1/k$ and $GI/M^Y/1/k$ queues and their application to the comparison for loss probabilities. *J. Appl. Probab.* **27**, 684–692.
- [17] Miyazawa, M., Sakuma, Y. and Yamaguchi, S. (2007). Asymptotic behaviors of the loss probability for a finite buffer queue with QBD structure. *Stoch. Models* **23**, 79–95.
- [18] Takagi, H. (1994). M/G/1//N queues with server vacations and exhaustive service. *Oper. Res.* **42**, 926–939.
- [19] Takine, T. (2004). Geometric and Subexponential Asymptotics of Markov Chains of M/G/1 Type. *Math. Operat. Res.* **29**, 624–648.
- [20] Wilf, H.S. (1994). *Generatingfunctionology*, 2nd ed. Academic Press, Boston, MA.
- [21] Willmot, G.E. (1988). A note on the equilibrium M/G/1 queue length. *J. Appl. Probab.* **25**, 228–231.

A Appendix

The following Definition A.1 and Theorem A.1 are taken from Definition VI.1 and Corollary VI.1 in [8], respectively, with a slight difference. Theorem A.1 is called Tauberian-like theorem in [14].

Definition A.1. Let $r \geq 1$. For given constants ϕ, \hat{r} with $\hat{r} > r$ and $0 < \beta < \frac{\pi}{2}$, define $\Delta(r, \beta, \hat{r})$ to be the following open domain

$$\Delta(r, \beta, \hat{r}) = \{z : |z| < \hat{r}, z \neq r, |\text{Arg}(z - r)| > \beta\}.$$

A domain is called a $\Delta(r)$ -domain if it is a $\Delta(r, \beta, \hat{r})$ for some \hat{r} and β . A function is said to be $\Delta(r)$ -analytic if it is analytic in some $\Delta(r)$ -domain.

Theorem A.1. Assume that $f(z)$ is $\Delta(1)$ -analytic and

$$f(z) \sim \frac{c}{(1-z)^\theta}, \quad \text{as } z \rightarrow 1, \quad z \in \Delta(1)$$

for some constant c and θ . Let f_n be the n th Taylor coefficient of $f(z)$. (i) If $\theta \notin \{0, -1, -2, \dots\}$, then

$$f_n \sim \frac{c}{\Gamma(\theta)} n^{\theta-1};$$

(ii) If $\theta \in \{0, -1, -2, \dots\}$, then

$$f_n = o(n^{\theta-1}).$$

The following theorems, referred to as Tauberian theorems, are taken from Section 5 of Chapter 13 in [7].

Theorem A.2. Let U be a measure on $[0, \infty]$ with Laplace transform $\omega(\tau) = \int_0^\infty e^{-\lambda x} U\{dx\}$. If L is a slowly varying function and $0 \leq \rho < \infty$, then the properties

$$\omega(\tau) \sim \tau^{-\rho} L\left(\frac{1}{\tau}\right), \quad \tau \rightarrow 0,$$

and

$$U(t) \sim \frac{1}{\Gamma(\rho+1)} t^\rho L(t), \quad t \rightarrow \infty,$$

are equivalent.

Theorem A.3. Let $q_n \geq 0$ and assume that

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n$$

converges for $0 \leq s < 1$. If L is a slowly varying function and $0 \leq \rho < \infty$, then the properties

$$Q(s) \sim \frac{1}{(1-s)^\rho} L\left(\frac{1}{1-s}\right), \quad s \rightarrow 1-, \tag{A.1}$$

and

$$q_0 + q_1 + \dots + q_{n-1} \sim \frac{1}{\Gamma(\rho+1)} n^\rho L(n), \quad n \rightarrow \infty,$$

are equivalent.